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Improved Shinnar–Le Roux Algorithm

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Selective excitation pulses are widely used in magnetic resonance imaging in order to excite predetermined slices of the body under examination. Such pulses are optimally designed by means of the Shinnar–Le Roux algorithm. In this paper, we show that under minimal assumptions, the complexity and computing cost of the original Shinnar–Le Roux algorithm can be drastically reduced. We further propose an improved version of the algorithm, involving only real quantities, which is both easier to implement and faster to execute, so it is suitable for implementation at the hardware level, in the context of a real-time fully digital magnetic resonance transceiver. © 2000 Academic Press

Key Words: **selective excitation; Shinnar–Le Roux algorithm; radiofrequency pulses; radiofrequency system; hardware implementation.**

1. INTRODUCTION

Magnetic resonance imaging offers a three-dimensional imaging modality. However, this is not always desirable, as 3D imaging has excessive requirements on experiment time, memory, and computing cost. Thus, covering the area of interest by means of a series of two-dimensional slices that offer the same diagnostic information at a lesser cost has been tried.

Reducing the dimensions being imaged is usually done by exciting only a slice of finite width and not the whole body under examination. This is done using a combination of a magnetic field gradient, called a selection gradient, and a specially designed radiofrequency pulse. Using a linear approximation (*1*), the regions excited are those with Larmor frequencies corresponding to the spectral content of the pulse.

The process of excitation, as described by the Bloch equations, is nonlinear, and the deviation from linearity, as well as the need for a more sophisticated approximation, becomes evident as the excitation angle increases. The latter is given in the Shinnar–Le Roux (*1, 2*) algorithm, which predicts the excitation angle with great precision.

The Shinnar–Le Roux algorithm deals with the one-dimensional excitation problem in the general case. We intend to show that under minimal assumptions the computing time and complexity can be drastically reduced.

Even though this may seem insignificant due to the evergrowing capacities of modern computers, it becomes significant when considering an integrated system thought to handle the calculation in real time. Indeed, real-time pulse generation can be useful in adjusting large flip angles at scan time, as in the case of nulling lipids using a fixed inversion time. CHESS pulses for water suppression are also a case that might call for real-time adjustment. It is also useful when designing multiband pulses if the exact slice locations are not known in advance. In general, real-time pulse generation is of value in cases of adaptive excitation, where it is needed to be able to adjust the excitation during scan time in order to gain focus on the item of interest.

2. THE SHINNAR–LE ROUX ALGORITHM

The Shinnar–Le Roux algorithm is based on a numerical solution of the Bloch equations. Three assumptions are made: First, the pulse envelope is piecewise constant in time for periods of Δt. This corresponds to the way pulses are generated in most commercial systems. Second, the effect of the radiofrequency pulse is the same for all proton populations. Third, the hard-pulse approximation: the rotation of the magnetization vector during the Δt period is considered to be small enough to be approximated as the outcome of two successive rotations, one around the radiofrequency pulse's axis and one around the axis of the superposition of the gradient and the static magnetic field. When these conditions are met, the excitation process can be approximated as the succession of the elementary rotations for each Δt.

Thus, the Shinnar–Le Roux algorithm treats the pulse as a series of small hard pulses, each one exhibiting an individual amplitude and phase. The outcome of the algorithm is stored in an array and retrieved in order to produce, by means of a complex I/Q modulation system, the desired radiofrequency pulse.

However, the degrees of freedom offered by the Shinnar–Le Roux algorithm have a cost: first, the requirement for simultaneous amplitude and phase modulation, a capacity that can lead to errors due to technical limitations, and second, the computational cost due to the complex numbers used throughout the algorithm.

Taking a closer look at the Shinnar–Le Roux algorithm, we see that it describes the movement–rotation of the magnetization by means of two FIR filters that represent its Cayley–Klein

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parameters (3). These are more adapted to describing quantum mechanical phenomena and also allow the easy synthesis of successive rotations. They are given as a 2×2 complex matrix $Q = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$, with the additional constrain $|\alpha|^2 + |\beta|^2 = 1$, thus requiring only three independent quantities. Perhaps the most intuitive approach to the Cayley–Klein parameters is through their relationship to the axis–angle representation of a rotation: If a rotation can be modeled as a rotation around an axis $N = (n_x, n_y, n_z)$ by an angle θ, then the matrix Q of the Cayley–Klein parameters is given by $Q = \cos\theta/2\ I - i \sin\theta/2\ (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)$, where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the Pauli spin matrices and I the 2×2 unitary matrix. The main advantage of the Cayley–Klein parameters is, however, that the matrix describing the synthesis of successive rotations can be calculated as the product of the matrices for each Δt.

Due to the hard-pulse approximation, the rotation for each Δt period can be described as

$$Q_j = \begin{pmatrix} C_j & -\bar{S}_j \\ S_j & C_j \end{pmatrix} \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}$$

where

$$\bullet C_j = \cos \frac{\gamma |B_{1,j}| \Delta t}{2}$$

$$\bullet S_j = i e^{i\angle B_{1,j}} \sin \frac{\gamma |B_{1,j}| \Delta t}{2} \qquad [1]$$

$$\bullet z = e^{i\gamma G x \Delta t}.$$

By means of space–state recursion, the Cayley–Klein parameters are given as

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} = z^{1/2} \begin{pmatrix} C_j & -\bar{S}_j \\ S_j & C_j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} \alpha_{j-1} \\ \beta_{j-1} \end{pmatrix},$$

and, by defining $A_j = z^{j/2}\alpha_j$ and $B_j = z^{j/2}\beta_j$,

$$\begin{pmatrix} A_j \\ B_j \end{pmatrix} = \begin{pmatrix} C_j & -\bar{S}_j z^{-1} \\ S_j & C_j z^{-1} \end{pmatrix} \begin{pmatrix} A_{j-1} \\ B_{j-1} \end{pmatrix}.$$

The parameters A and B are polynomials of z of grade $j - 1$. Their dependence on z expresses the spatial differentiation of the pulse effect due to the selection gradient.

Design using the Shinnar–Le Roux algorithm is based on the inversion of the above recursion formula, given by

$$\begin{pmatrix} A_{j-1} \\ B_{j-1} \end{pmatrix} = \begin{pmatrix} C_j & \bar{S}_j \\ -S_j z & C_j z \end{pmatrix} \begin{pmatrix} A_j \\ B_j \end{pmatrix} = \begin{pmatrix} C_j A_j + \bar{S}_j B_j \\ z(-S_j A_j + C_j B_j) \end{pmatrix}. \qquad [2]$$

It can be proven that

$$\frac{B_{j,0}}{A_{j,0}} = \frac{S_j}{C_j} = \frac{i e^{i\theta_j} \sin \frac{\phi_j}{2}}{\cos \frac{\phi_j}{2}}$$

and that the characteristics of the radiofrequency waveform can be given as

- $\phi_j = 2\arctan\left|\frac{B_{j,0}}{A_{j,0}}\right|$, the total rotation angle during the jth Δt period;
- $\theta = \angle\left(-i \frac{B_{j,0}}{A_{j,0}}\right)$, the phase of the radiofrequency pulse;
- $B_{1,j} = \frac{1}{\gamma\Delta t} \phi_j e^{i\theta_j}$, the waveform of the radiofrequency complex envelope.

3. DEGREES OF FREEDOM IN THE MODULATION PROCESS: DISCARDING PHASE INFORMATION

The results of the Shinnar–Le Roux algorithm, as described above, have two degrees of freedom, corresponding to amplitude ($|B_1|$) and phase (θ) modulation. What will be proven is that if the excitation characteristics are symmetric around the point for which the gradient amplitude is zero, i.e., the point for which the Larmor frequency equals the carrier frequency, a single degree of freedom suffices.

The term "excitation characteristics" refers to the form of the Cayley–Klein parameters as a function of the (spatial) selection gradient axis. The Cayley–Klein parameters describe the effect of excitation independently from the initial condition; that is to say they describe the excitation process and not its result. Thus, the symmetry refers to the rotation itself and not necessarily to the magnetization distribution after the radiofrequency pulse.

In the designing process, the parameters A and B are designed as FIR filters whose frequency response approximates the desired excitation characteristics. After B is designed, it is multiplied by a complex factor so that its phase accounts for the orientation of the radiofrequency pulse's axis on the xy plane of the rotating frame.

Since the filters A and B are by definition proportional to the Cayley–Klein parameters of the pulse, the effect of these symmetry constrains is that they are conjugate symmetric, i.e., $A(\omega) = -\overline{A(-\omega)}$ and $B(\omega) = -\overline{B(-\omega)}$. In this case, it is known from digital signal processing that FIR filters with conjugate symmetric frequency response have real coefficients (4), and thus $A_k(z) = \sum_{j=0}^{k-1} a_{k,j} z^{-j}$ and $B_k(z) = \sum_{j=0}^{k-1} b_{k,j} z^{-j}$, where $a_{k,j}, b_{k,j} \in R$. In order to account for the pulse orientation factor in the rotating frame, we consider A and B polynomials in the form $A_k(z) = m_1 \sum_{j=0}^{k-1} a_{k,j} z^{-j}$ and $B_k(z) = m_2 \sum_{j=0}^{k-1} b_{k,j} z^{-j}$, where $m_1, m_2 \in C$ with $|m_1| = |m_2| = 1$.

Considering the above, the orientation angle of the radiofrequency pulse's axis is calculated as

$$\theta_k = \angle\left(-i\,\frac{m_2 b_{k,0}}{m_1 a_{k,0}}\right) = \theta + \angle\left(\frac{b_{k,0}}{a_{k,0}}\right).$$

However, since $a_{k,0}$ and $b_{k,0}$ are real, the second term can only calculate to 0° or 180°. Thus, its effect can be considered a change in the sign of the envelope and can thus be coded as amplitude and not necessarily phase information.

By substitution in Eq. [1], we get

$$S_k = i e^{\,i\{\angle(-i(m_2/m_1)) + \angle(b_{k,0}/a_{k,0})\}} \sin\frac{\phi_k}{2}.$$

However, $|-i(m_2/m_1)| = 1$, and thus $e^{i\{\angle(-i(m_2/m_1))\}} = -i(m_2/m_1)$. Furthermore, since $a_{k,0}$ and $b_{k,0}$ are real, $e^{i\{\angle(b_{k,0}/a_{k,0})\}} = \mathrm{sgn}(b_{k,0}/a_{k,0})$. Thus, $S_k = i(-i(m_2/m_1))\mathrm{sgn}(b_{k,0}/a_{k,0})\sin\phi_k/2 = m_2/m_1\, s_k$, where $s_k = \mathrm{sgn}(b_{k,0}/a_{k,0})\sin\phi_k/2 \in R$. If we go further and code the sign information into the amplitude representation by assigning $\phi_k = 2\tan^{-1}(b_{k,0}/a_{k,0})$, then $s_k = \sin\phi_k/2$. By definition, $C_k \in R$. C_k remains unaffected by the sign change in ϕ_k, since the cosine function exhibits even symmetry.

Substituting in the recursion formula of Eq. [2], we get

$$\begin{aligned} A_{k-1}(z) &= C_k m_1 \sum_j a_{k,j} z^{-j} + \overline{\frac{m_2}{m_1}}\, s_k m_2 \sum_j b_{k,j} z^{-j} \\ &= m_1\left(C_k \sum_j a_{k,j} z^{-j} + s_k \frac{|m_2|^2}{|m_1|^2} \sum_j b_{k,j} z^{-j}\right) \\ &= m_1 \sum_j a_{k-1,j} z^{-j} \qquad a_{k,j},\ a_{k-1,j} \in R \end{aligned}$$

and

$$\begin{aligned} B_{k-1}(z) &= z\left(-\frac{m_2}{m_1} s_k m_1 \sum_j a_{k,j} z^{-j} + C_k m_2 \sum_j b_{k,j} z^{-j}\right) \\ &= m_2 \sum_j b_{k-1,j} z^{-j} \qquad b_{k,j},\ b_{k-1,j} \in R. \end{aligned}$$

Thus, having coded the sign information into the amplitude representation (angle ϕ), the radiofrequency axis for the $(k-1)$th step of the recursion calculates to

$$\theta_{k-1} = \angle\left(-i\,\frac{B_{k-1,0}}{A_{k-1,0}} \cdot \frac{a_{k,0}}{b_{k,0}}\right) = \angle\left(-i\,\frac{m_2}{m_1}\right) = \theta_k.$$

By means of induction we derive that the radiofrequency pulse's axis remains constant throughout the recursion. It should be noted, however, that due to the different definition of the angle ϕ the amplitude can change sign, and so it belongs to R and should not be considered an element of R^+.

Given the above, the results of the design process are shown to be a constant angle θ describing the orientation of the radiofrequency pulse's axis and a varying, thus selective, envelope, described by $B_1(t)$.

The angle θ describes the orientation of the radiofrequency pulse on the xy plane. However, it is known that the axes x and y of the rotating frame are set by the phase of the first radiofrequency pulse in every experiment. Furthermore, most imaging sequences rely on pulses having a constant orientation, by definition among the x axis, or at the most they use quadrature angles, like in the Carr–Purcell–Meiboom–Gill spin-echo sequence (*5*). Thus, for the carrier generation we need a sine generator and for special cases its cosine.

This carrier is used to modulate, by means of double-sideband, suppressed carrier modulation, the function $B_1(t)$. Spectrally, this can be derived from the fact that in the design process we take both positive and negative frequencies into account. Thus, there is no issue of suppressing a sideband. Technically, this leads to a simplified radiofrequency generation system.

4. THE SIMPLIFIED SHINNAR–LE ROUX ALGORITHM

A direct consequence of the constant radiofrequency pulse's axis is the simplification of the Shinnar–Le Roux algorithm, since we no longer need to calculate θ at every recursion step. What follows is a simplified version of the Shinnar–Le Roux algorithm:

1. The inputs are given in the form $A_n(z) = \sum_{j=0}^{n-1} a_j z^{-j}$ and $B_n(z) = \sum_{j=0}^{n-1} b_j z^{-j}$, where $a_j, b_j \in R$. A second, independent input is the ratio m_2/m_1, which is needed in order to calculate the axis angle θ.
2. The angle $\phi_k = 2\arctan(b_{k,0}/a_{k,0})$ is calculated for every recursion step. This definition of ϕ_k is different from the one in the original Shinnar–Le Roux algorithm, in order to incorporate the sign information, thus transferring in ϕ_k the only freedom of change of θ. Following quantities are calculated from ϕ_k:

- $s_k = \sin\dfrac{\phi_k}{2}$;
- $\mathrm{C}_k = \cos\dfrac{\phi_k}{2}$;
- $\mathrm{B}_{1,k} = \dfrac{1}{\gamma\Delta t}\,\phi_k$.

TABLE 1
Duration of the Design Process (in ms/Pentium II 350-MHz Computer)

Filter length	Simplified SLR backrecursion	Simplified SLR last two stages	Conventional SLR backrecursion	Conventional SLR last two stages	Backrecursion gain ratio	Last two stages gain ratio
61	0.65	5.27	7.69	12.53	11.8	2.4
101	1.59	11.65	20.05	30.71	12.6	2.6
151	3.41	25.10	43.83	66.90	12.8	2.7
201	5.77	27.57	76.79	100.74	13.3	3.6
251	8.84	30.59	118.81	143.90	13.4	4.7
301	12.53	59.92	169.94	222.45	13.5	3.7

This time, B_1 corresponds to the pulse's signed amplitude. Combined with the axis information coded in the angle θ we have the full description of the pulse.

3. Now, by using the equations $A_{k-1}(z) = m_1(C_k \sum_j a_{k,j}z^{-j} + s_k \sum_j b_{k,j}z^{-j})$ and $B_{k-1}(z) = m_2 z(-s_k \sum_j a_{k,j}z^{-j} + C_k \sum_j b_{k,j}z^{-j})$ and ignoring m_1 and m_2, we can calculate the polynomials $A_{k-1}(z)$ and $B_{k-1}(z)$ to be used in the next recursion step. Since $a_{k-1,j}$ depends on $a_{k,j}$ and $b_{k,j}$ while $b_{k-1,j}$ depends on $a_{k,j+1}$ and $b_{k,j+1}$, the calculation can be performed in place: We first calculate and substitute $a_{k-1,j}$ and then $b_{k-1,j}$, thus performing a memory-efficient procedure.

What should be noted is that all of the quantities involved in the recursion are real, thus reducing the requirements both in memory and in computing cost and complexity. The memory requirements for an N-point pulse are three real arrays of length N (two for the filters and one for the outcome) and four accumulators for s, C, ϕ, and θ. Furthermore, the main part of the computational cost of the original Shinnar–Le Roux algorithm is due to the repeated θ calculation as well as to the complex multiplications, which are equivalent to four real multiplications each. Thus our version is significantly faster than the original.

Pulse design using the Shinnar–Le Roux algorithm goes through three major steps. The first step involves designing the polynomial B, which may use any of the FIR filter design techniques that are widely used in digital signal processing. The time needed in this stage varies significantly according to the method selected. However, if we only want to adjust the flip angle, the form of B can be calculated only once and be retrieved from memory to be adjusted by a multiplication by $\sin \phi/2$, where ϕ is the total flip angle. The calculation of the polynomial A takes place in the second step. This is usually a minimum-phase filter whose calculation involves complex cepstrum techniques and Fourier transform calculations. The third step involves the backrecursion discussed above.

In order to estimate the acceleration of the design process, we compared the time needed for the backrecursion and the two final design stages using the original and our simplified version of the Shinnar–Le Roux algorithm. Results are shown in Table 1. Timings are given in milliseconds on a Pentium II 350-MHz-based computer. What we derive is that the time needed for the backrecursion is one order of magnitude less than in the original version, while the last two stages are two to four times faster, depending on the relation between the filter length and the size of the FFT used.

Furthermore, even though the results of the two algorithms are expected to be identical, the reduction of computing is expected to minimize round-off errors during the calculation and therefore provide more accurate results.

5. CONCLUSIONS

It has been shown that, for excitation characteristics that are symmetric around the beginning of the gradient axis, a single degree of freedom suffices for the modulation of the radiofrequency excitation pulse. This symmetry refers to the location of the zeros of A and B and not to any phase offsets introduced by manipulating the orientation of the radiofrequency pulse. This single degree of freedom is due to the fact that the phase axis of the desired pulse remains constant. A constant phase factor can be introduced to account for the orientation of this axis on the xy plane of the rotating frame. However, as the orientation of the x and y axes is defined by the carrier phase of the initial pulse and remains constant throughout the experiment for most imaging methods, the absolute radiofrequency phase should be of no importance as long as we keep a steady carrier source as a reference.

This result has a significant impact on the pulse generation system. It is known that for a signal with arbitrarily varying phase, errors in the generation of the modulated signal cannot be avoided. What has been shown is that it suffices once the phase of the carrier to adjust and treat the rest of the production as an amplitude modulation process. Given the usual orientation of the radiofrequency pulses in the rotating frame, i.e., along the x or in special cases along the y axis, the most required of the modulator is to produce the quadrature angles, that is to say 0°, 90°, 180°, and 270°. These angles can be achieved easily and with greater precision.

The assumption of symmetrical excitation characteristics holds for the vast majority of one-dimensional selective excitation problems. In most cases a uniform distribution of the magnetization in the excited region is desired, which is practically equivalent to the symmetry conditions assumed in our algorithm. Even in the case of minimum- and maximum-phase pulses, the design process is based on real and thus symmetrical polynomials.

Furthermore, a simplified version of the Shinnar–Le Roux algorithm has been presented. In this version, by taking advantage of the constant θ angle, all of the quantities involved are real, thus reducing the memory requirements, the computational cost, and the complexity of the original algorithm.

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